

AMERICAN UNIVERSITY OF THE MIDDLE EAST
MA265 – Linear Algebra, Online Spring 2, 2020
Final Assessment

Student Name:

Student ID:

Section: O4

Show your full and complete work.

Q1. Let $\mathbf{A} = \begin{bmatrix} 6 & -1 \\ 50 & -9 \end{bmatrix}$ be a 2×2 matrix.

(a) Find the characteristic polynomial of the matrix $\mathbf{A}$. (5 pts)

(b) Find all eigenvalues and associated eigenvectors of the matrix $\mathbf{A}$. (10 pts)

(c) If λ is an eigenvalue of $\mathbf{A}$, what do you think it would be the eigenvalue of the matrix $-8\mathbf{A}$? (Justify your answer) (5 pts)

Q2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 6 & 0 & 0 \\ 3 & -1 & -2 \\ -7 & 0 & 5 \end{bmatrix}$.

(a) Find all eigenvalues of the matrix $\mathbf{A}$. **(7 pts)**

(b) Find all eigenvectors of the matrix $\mathbf{A}$. **(8 pts)**

(c) Do you think that the set of the eigenvectors of $\mathbf{A}$ is a basis for the vector space $\mathbb{R}^3$? (Justify your answer) **(5 pts)**

Q3. Consider the vector space P_2 consisting of all polynomials of degree at most two together with the zero polynomial. Let $S = \{p_1(t), p_2(t)\}$ be a set of polynomials in P_2 where:

$$p_1(t) = -t^2 - 7, \quad p_2(t) = -2t^2 + 2t - 7$$

(a) Determine whether the set $S = \{p_1(t), p_2(t)\}$ is linearly independent in P_2 ? Provide a clear justification for your solution. **(8 pts)**

(b) Determine whether the set $S = \{p_1(t), p_2(t)\}$ spans the vector space P_2 ? Provide a clear justification for your solution. **(7 pts)**

(c) Is it possible to find a polynomial $p_3(t)$ in P_2 so that $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for the vector space P_2 ? (Justify your answer). **(5 pts)**

Q4. Let $L : R^2 \longrightarrow R^3$ be a transformation defined by $L \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) = \begin{bmatrix} 7u_2 - u_1 \\ u_1 - u_2 \\ -3u_2 \end{bmatrix}$.

(a) Show that L is a linear transformation. (**8 pts**)

(b) Find the standard matrix $\mathbf{A}$ of L , and find $L \left(\begin{bmatrix} -4 \\ 3 \end{bmatrix} \right)$ using the matrix $\mathbf{A}$. (**7 pts**)

(c) Do you think that any transformation $T : R^2 \longrightarrow R^3$ is linear? (Justify your answer).(**5 pts**)

Q5. Consider the square matrix $\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -4 & 1 \end{bmatrix}$.

(a) Show that the characteristic polynomial of $\mathbf{A}$ is: $p(\lambda) = \lambda^2 - 3\lambda - 6$. (5 pts)

(b) Compute the matrix $B = \mathbf{A}^2 - 3\mathbf{A} - 6I_2$. (5 pts)

(c) Show that $\mathbf{A}^2 - 3\mathbf{A} = 6I_2$ for the given matrix $\mathbf{A}$. (5 pts)

(d) Is it possible to use the equation $\mathbf{A}^2 - 3\mathbf{A} = 6I_2$ to find the inverse of the given matrix $\mathbf{A}$?
(Justify your answer) (5 pts)

#MA 265 (FA) (04) #

Q1: $A = \begin{pmatrix} 6 & -1 \\ 50 & -9 \end{pmatrix}$

(a) $A - \lambda I_2 = \begin{pmatrix} 6 & -1 \\ 50 & -9 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 6-\lambda & -1 \\ 50 & -9-\lambda \end{pmatrix}$

$p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 6-\lambda & -1 \\ 50 & -9-\lambda \end{vmatrix}$

$= (6-\lambda)(-9-\lambda) - (-50)$

$= -54 - 6\lambda + 9\lambda + \lambda^2 + 50$

$p(\lambda) = \lambda^2 + 3\lambda - 4 \rightarrow \text{characteristic polynomial}$

(b) put $p(\lambda) = 0 \Rightarrow \lambda^2 + 3\lambda - 4 = 0$

$(\lambda + 4)(\lambda - 1) = 0$

$\lambda_1 = 1$

$\lambda_2 = -4$

These are the eigenvalues

to find eigen vectors associated with

eigen values, we have to solve $\Rightarrow$

Q1
Ex 1

The Non trivial homogenous system

$$(A - \lambda I_2) V = 0 \Leftrightarrow \left(\begin{array}{cc|c} 6-\lambda & -1 & 0 \\ 50 & -9-\lambda & 0 \end{array} \right)$$

For $\lambda_1 = 1$: $\left(\begin{array}{cc|c} 5 & -1 & 0 \\ 50 & -10 & 0 \end{array} \right) R_1/5 \rightarrow R_1$

$$\left(\begin{array}{cc|c} 1 & -1/5 & 0 \\ 50 & -10 & 0 \end{array} \right) -50R_1 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{cc|c} 1 & -1/5 & 0 \\ 0 & 0 & 0 \end{array} \right) \text{ Non trivial solution}$$

$$V_1 - 1/5 V_2 = 0$$

If we take $V_2 = t, t \in \mathbb{R} \Rightarrow V_1 = t/5$

$$\therefore V = \begin{pmatrix} t/5 \\ t \end{pmatrix} = \begin{pmatrix} 1/5 \\ 1 \end{pmatrix} t \text{ is an eigen}$$

vector with $\lambda_1 = 1$



Q1

For $\lambda = -4$: $\left(\begin{array}{cc|c} 10 & -1 & 0 \\ 50 & -5 & 0 \end{array} \right) R_1/10 \rightarrow R_1$

$\left(\begin{array}{cc|c} 1 & -1/10 & 0 \\ 50 & -5 & 0 \end{array} \right) -50R_1 + R_2 \rightarrow R_2 \left(\begin{array}{cc|c} 1 & -1/10 & 0 \\ 0 & 0 & 0 \end{array} \right)$
 No trivial solution

$$V_1 - \frac{1}{10} V_2 = 0$$

∴ we take $V_2 = t, t \in \mathbb{R} \Rightarrow V_1 = \frac{1}{10} t$

$$\therefore V = \begin{pmatrix} \frac{1}{10} t \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{10} \\ 1 \end{pmatrix} t \text{ is}$$

eigenvector with $\lambda_2 = -4$

Q IF λ is an eigen value of A then the eigen value for $-8A$ is -8λ

we know that

$$(A V_A = \lambda_A V_A) \quad * -8$$

$$-8 A V_A = -8 \lambda_A V_A$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$B \cdot V_B = \lambda_B V_B$$

, so the eigen value will be $\boxed{\lambda_B = -8 \lambda_A}$

Q2

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 3 & -1 & -2 \\ -7 & 0 & 5 \end{pmatrix}$$

(a)

$$A - \lambda I_3 = \begin{pmatrix} 6 & 0 & 0 \\ 3 & -1 & -2 \\ -7 & 0 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 6-\lambda & 0 & 0 \\ 3 & -1-\lambda & -2 \\ -7 & 0 & 5-\lambda \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 6-\lambda & 0 & 0 \\ 3 & -1-\lambda & -2 \\ -7 & 0 & 5-\lambda \end{vmatrix}$$

$$= (6-\lambda) \begin{vmatrix} -1-\lambda & -2 \\ 0 & 5-\lambda \end{vmatrix} = (6-\lambda) [(-1-\lambda)(5-\lambda) - 0]$$

$$P(\lambda) = (6-\lambda)(-1-\lambda)(5-\lambda)$$

Characteristic polynomial

$$\text{put } P(\lambda) = 0 \Rightarrow (6-\lambda)(-1-\lambda)(5-\lambda) = 0$$

$$\lambda_1 = 6$$

$$\lambda_2 = 5$$

$$\lambda_3 = -1$$

These are the
eigen values.

Q2
b

to Find eigen vectors associated with eigenvalue, we have to solve the Non trivial homogenous system.

$$(A - \lambda I_3) V = 0 \Leftrightarrow \left(\begin{array}{ccc|c} 6-\lambda & 0 & 0 & 0 \\ 3 & -1-\lambda & -2 & 0 \\ -7 & 0 & 5-\lambda & 0 \end{array} \right)$$

For $\lambda_1 = 6$: $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 3 & -7 & -2 & 0 \\ -7 & 0 & -1 & 0 \end{array} \right) R_1 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|c} -7 & 0 & -1 & 0 \\ 3 & -7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) R_1 / -7 \rightarrow R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/7 & 0 \\ 3 & -7 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) -3R_1 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/7 & 0 \\ 0 & -7 & -17/7 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) -1/7 R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1/7 & 0 \\ 0 & 1 & 17/49 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ Non trivial solution}$$

$$V_1 + 1/7 V_3 = 0$$

$$V_2 + 17/49 V_3 = 0$$



Q2
b

If we take $V_3 = t, t \in \mathbb{R}$

$$\therefore V_1 = -\frac{1}{7}t, V_2 = -\frac{17}{49}t$$

$$\therefore V = \begin{bmatrix} -\frac{t}{7} \\ -\frac{17}{49}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} \\ -\frac{17}{49} \\ 1 \end{bmatrix} t \text{ is eigen vector}$$

For $\lambda_2 = 5$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & -6 & -2 & 0 \\ -7 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} -3R_1 + R_2 \rightarrow R_2 \\ 7R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) R_2 / -6 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ Nontrivial solution}$$

$$V_1 = 0, V_2 + \frac{1}{3}V_3 = 0$$

$$\text{take } V_3 = t, t \in \mathbb{R} \Rightarrow V_2 = -\frac{1}{3}t$$

$$\therefore V = \begin{pmatrix} 0 \\ -\frac{1}{3}t \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix} \text{ is eigen vector}$$

Q₂
b

For $\lambda_3 = -1$: $\left(\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ -7 & 0 & 6 & 0 \end{array} \right) R_1/7 \rightarrow R_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ -7 & 0 & 6 & 0 \end{array} \right) \begin{array}{l} -3R_1 + R_2 \rightarrow R_2 \\ 7R_1 + R_3 \rightarrow R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 6 & 0 \end{array} \right) R_2/-2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 \end{array} \right) -6R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ No non trivial solution.}$$

$$V_1 = 0, V_3 = 0, V_2 = t, t \in \mathbb{R}$$

$$\therefore V = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is}$$

eigenvector.

Q₂
C

Yes, the set of the eigenvectors of A is a basis for the vector space $\mathbb{R}^3$ because three distinct eigenvalues associated three eigenvectors are linearly independent

$$\det(P) = \begin{vmatrix} -1/7 & 0 & 0 \\ -17/49 & -1/3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\det(P) = -1/7 \begin{vmatrix} -1/3 & 1 \\ 1 & 0 \end{vmatrix} = -1/7 (0 - 1) = 1/7 \neq 0$$

∴ the set of eigenvectors $\left\{ \left(-\frac{1}{7}, -\frac{17}{49}, 1\right), \left(0, -\frac{1}{3}, 1\right), (0, 1, 0) \right\}$

is a basis for $\mathbb{R}^3$.

Q3:

(a) $C_1 p_1(t) + C_2 p_2(t) = \mathbf{0}_V$

$$C_1 (-t^2 - 7) + C_2 (-2t^2 + 2t - 7) = (0, 0, 0)$$

$$\begin{array}{cc} C_1 & C_2 \\ \left(\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 2 & 0 \\ -7 & -7 & 0 \end{array} \right) & \xrightarrow{-R_1} R_1 & \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -7 & -7 & 0 \end{array} \right) \end{array} \quad 7R_1 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 7 & 0 \end{array} \right) \xrightarrow{R_2/2} R_2 \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 0 \end{array} \right) \quad -7R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \text{this system has trivial solution with } C_1 = C_2 = 0$$

$\therefore$ the set $S = \{p_1(t), p_2(t)\}$ is linearly independent in P_2

(b) $C_1 p_1(t) + C_2 p_2(t) = p_2$

$$C_1 (-t^2 - 7) + C_2 (-2t^2 + 2t - 7) = at^2 + bt + c$$

$$\begin{array}{cc} C_1 & C_2 \\ \left(\begin{array}{cc|c} -1 & -2 & a \\ 0 & 2 & b \\ -7 & -7 & c \end{array} \right) & \xrightarrow{-R_1} R_1 & \left(\begin{array}{cc|c} 1 & 2 & -a \\ 0 & 2 & b \\ -7 & -7 & c \end{array} \right) \end{array} \quad 7R_1 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{cc|c} 1 & 2 & -a \\ 0 & 2 & b \\ 0 & 7 & -7a + c \end{array} \right) \xrightarrow{R_2/2} R_2 \left(\begin{array}{cc|c} 1 & 2 & -a \\ 0 & 1 & b/2 \\ 0 & 7 & -7a + c \end{array} \right)$$

Q3

$$(b) \left(\begin{array}{cc|c} 1 & 2 & -a \\ 0 & 1 & b/2 \\ 0 & 7 & -7a+c \end{array} \right) \xrightarrow{-7R_2+R_3} R_3$$

$$\left(\begin{array}{cc|c} 1 & 2 & -a \\ 0 & 1 & b/2 \\ 0 & 0 & -7/2 b - 7a + c \end{array} \right)$$

It follows that the system is consistent if and only if a, b, c satisfy $-7/2 b - 7a + c = 0$

That means Not every $v = (a, b, c)$ in P_2 belongs to the span.

Hence the set

$S = \{p_1(t), p_2(t)\}$ do not

span the vector space P_2

Q3

C

Yes it is possible, to find $p_3(t)$ in P_2 such that $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for the vector space P_2 .

We must add e_1, e_2, e_3 and look

for linearly independent vectors.

$$\begin{array}{ccccc} p_1 & p_2 & e_1 & e_2 & e_3 \\ \left(\begin{array}{ccccc} -1 & -2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -7 & -7 & 0 & 0 & 1 \end{array} \right) & -R_1 \rightarrow R_1 \end{array}$$

$$\left(\begin{array}{ccccc} 1 & 2 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -7 & -7 & 0 & 0 & 1 \end{array} \right) 7R_1 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccccc} 1 & 2 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 7 & -7 & 0 & 1 \end{array} \right) R_2/2 \rightarrow R_2$$

$$\left(\begin{array}{ccccc} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 7 & -7 & 0 & 1 \end{array} \right) -7R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccccc} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & -7 & -7/2 & 1 \end{array} \right) R_3 / -7 \rightarrow R_3 \Rightarrow$$



$$\begin{pmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -1/7 \end{pmatrix}$$

so the set of $\{p_1(t), p_2(t), e_1\}$ is
a basis for P_2

we must add $p_3(t) = t^2$

$$so S = \{-t^2 - 7, -2t^2 + 2t - 7, t^2\}$$

is a basis for P_2 .

Q4

$$L: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 7u_2 - u_1 \\ u_1 - u_2 \\ -3u_2 \end{bmatrix}$$

(a) let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$L(u+v) = L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 7(u_2+v_2) - (u_1+v_1) \\ (u_1+v_1) - (u_2+v_2) \\ -3(u_2+v_2) \end{bmatrix}$$

$$= \begin{bmatrix} 7u_2 + 7v_2 - u_1 - v_1 \\ u_1 + v_1 - u_2 - v_2 \\ -3u_2 - 3v_2 \end{bmatrix}$$

$$= \begin{bmatrix} (7u_2 - u_1) + (7v_2 - v_1) \\ (u_1 - u_2) + (v_1 - v_2) \\ -3u_2 - 3v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 7u_2 - u_1 \\ u_1 - u_2 \\ -3u_2 \end{bmatrix} + \begin{bmatrix} 7v_2 - v_1 \\ v_1 - v_2 \\ -3v_2 \end{bmatrix}$$

$$= L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + L\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

$$L(u+v) = L(u) + L(v) \quad \text{—————} \boxed{1}$$

Q4

$$(a) \quad k \in \mathbb{R}, \quad L(ku) = L\left(k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) \quad 1$$

$$= L\left(\begin{bmatrix} ku_1 \\ ku_2 \end{bmatrix}\right) = \begin{bmatrix} 7ku_2 - ku_1 \\ ku_1 - ku_2 \\ -3ku_2 \end{bmatrix}$$

$$= \begin{bmatrix} k(7u_2 - u_1) \\ k(u_1 - u_2) \\ -3ku_2 \end{bmatrix}$$

$$= k \begin{bmatrix} 7u_2 - u_1 \\ u_1 - u_2 \\ -3u_2 \end{bmatrix}$$

$$L(ku) = k L\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = k L(u) \quad \text{--- (2)}$$

From 1, 2

$\therefore L$ is a linear transformation.

Q4
b) The standard basis for $\mathbb{R}^2$ is $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$L(e_1) = L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 7(0) - 1 \\ 1 - 0 \\ -3(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$L(e_2) = L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 7(1) - 0 \\ 0 - 1 \\ -3(1) \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -3 \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} L(e_1) & L(e_2) \\ -1 & 7 \\ 1 & -1 \\ 0 & -3 \end{pmatrix}$$

$$\begin{aligned} L\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right) &= A \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 1 & -1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -4 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 + 21 \\ -4 + -3 \\ 0 + -9 \end{pmatrix} = \begin{pmatrix} 25 \\ -7 \\ -9 \end{pmatrix} \end{aligned}$$

c) NO,

$$\text{if } T: \mathbb{R}^2 \rightarrow \mathbb{R}^3, T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_2 \\ 4 \end{pmatrix}$$

$$\text{let } u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$T(u+v) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ 4 \end{pmatrix}$$

$$T(u) + T(v) = \begin{pmatrix} u_1 \\ u_2 \\ 4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ 4 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ 8 \end{pmatrix}$$

$\therefore T(u+v) \neq T(u) + T(v) \Rightarrow \text{Not Linear.}$

Q5

$$A = \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$(a) A - \lambda I_2 = \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 2-\lambda & -2 \\ -4 & 1-\lambda \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & -2 \\ -4 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda) - 8$$

$$= 2 - 2\lambda - \lambda + \lambda^2 - 8$$

$P(\lambda) = \lambda^2 - 3\lambda - 6$ is the
Characteristic polynomial.

$$(b) B = A^2 - 3A - 6I_2$$

$$= \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix} - 3 \begin{pmatrix} 2 & -2 \\ -4 & 1 \end{pmatrix} - 6 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+8 & -4+2 \\ -8-4 & 8+1 \end{pmatrix} - \begin{pmatrix} 6 & -6 \\ -12 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & -6 \\ -12 & 9 \end{pmatrix} - \begin{pmatrix} 6 & -6 \\ -12 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O_2$$

Q5
(c)

From part (b):

$$p(A) = 0$$

$$A^2 - 3A - 6I_2 = 0$$

$$A^2 - 3A - \underline{6I_2} + \underline{6I_2} = 0 + 6I_2$$

$$\therefore A^2 - 3A = 6I_2$$

(d) Yes it is possible

$$A^2 - 3A = 6I_2$$

(multiply by $\frac{1}{6}$)
both sides

$$\frac{1}{6} (A^2 - 3A) = I_2$$

$$\frac{1}{6} (A \underbrace{(A - 3I_2)}) = I_2$$

$$A \left(\underbrace{\frac{1}{6} (A - 3I_2)}_{\Downarrow B} \right) = I_2$$

$$A \cdot B = I_2 \Leftrightarrow B = A^{-1}$$

$$\therefore A^{-1} = \frac{1}{6} (A - 3I_2) \quad \text{, hint } \boxed{A \cdot A^{-1} = I_n}$$